LECTURES on COLLISIONAL DYNAMICS:

1. RELEVANT TIMESCALES
COLLISIONAL/COLLISIONLESS?

Collisional systems are systems where interactions between particles are EFFICIENT with respect to the lifetime of the system.
Collisionless systems are systems where interactions are negligible.

When is a system collisional/collisionless?

RELAXATION TIMESCALE

Gravity is a LONG-RANGE force → cumulative influence on each star/body of distant stars/bodies is important: often more important than influence of close stars/bodies.

Let us consider a IDEALIZED galaxy of N identical stars with mass $m$, size $R$ and uniform density.
Let us focus on a single star that crosses the system.

How long does it take for this star to change its initial velocity completely?, i.e. by

$$\frac{\delta \vec{v}_\perp}{\vec{v}} \sim 1$$
Let us assume that our test star passes close to a field star at relative velocity $v$ and impact parameter $b$.

The test star and the perturber interact with a force

$$F_\perp = \frac{G m^2}{r^3} \quad r_\perp = \frac{G m^2}{r^2} \cos \theta$$

$$= \frac{G m^2 b}{(x^2 + b^2)^{3/2}} = \frac{G m^2}{b^2} \left[ 1 + \left( \frac{v t}{b} \right)^2 \right]^{3/2}$$
From Newton's second law \( m \ddot{v}_\perp = F_\perp \)

we get that the perturbation of the velocity integrated over one entire encounter is

\[
\delta v_\perp = \int_{-\infty}^{+\infty} \frac{F_\perp}{m} dt = \frac{G m}{b^2} \int_{-\infty}^{+\infty} \frac{dt}{\left[1 + \left(\frac{v t}{b}\right)^2\right]^{3/2}}
\]

\[
= \frac{G m}{b v} \int_{-\infty}^{+\infty} \frac{ds}{(1 + s^2)^{3/2}} = \frac{2 G m}{b v}
\]

\[
dt = \frac{b}{v} d\left(\frac{v t}{b}\right)
\]

\[
2 \left(\frac{G m}{b^2}\right) \left(\frac{b}{v}\right)
\]

Force at closest approach  
Force duration
Now we account for all the particles in the system.

Surface density of stars in idealized galaxy:

\[
\frac{N}{\pi R^2}
\]

Number of interactions per unit element:

\[
\delta n = \frac{N}{\pi R^2} 2\pi b \, db
\]

We define

\[
\delta v^2_{\text{TOT}} = \delta n \, \delta v^2_{\perp} = \frac{2N}{R^2} \left( \frac{2Gm}{bv} \right)^2 b \, db
\]

And we integrate over all the possible impact parameters...
And we integrate over all the possible impact parameters...

\[ \delta v_{TOT}^2 = 8 N \left( \frac{G m}{R v} \right)^2 \int_{b_{\text{min}}}^{R} \frac{db}{b} \]

\[ \delta v_{TOT}^2 = 8 N \left( \frac{G m}{R v} \right)^2 \ln \frac{R}{b_{\text{min}}} \]

* low integration limit: smallest \( b \) to avoid close encounter \( \delta v_\perp \sim v \)

\[ v = \frac{2 G m}{b_{\text{min}} v} \quad \implies \quad b_{\text{min}} = \frac{2 G m}{v^2} \]

* top integration limit: size \( R \) of the system
And we integrate over all the possible impact parameters...

\[ \delta v_{\text{TOT}}^2 = 8N \left( \frac{Gm}{Rv} \right)^2 \ln \left( \frac{Rv^2}{2Gm} \right) \]

Typical speed of a star in a virialized system

\[ N \, m \, v^2 = \frac{G \left( N \, m \right)^2}{R} \quad \Rightarrow \quad v^2 = \frac{G}{N \, m} \frac{R}{\cancel{\frac{N \, m}{N \, m}}} \]

Replacing \( v \)

\[ \frac{\delta v_{\text{TOT}}^2}{v^2} = \frac{8 \ln N}{N} \]
Number of crossings of the system for which

\[ \frac{\delta v^2_{\text{TOT}}}{v^2} = 1 \]

\[ n_{\text{cross}} = \frac{N}{8 \ln N} \]

**CROSSING TIME** = time needed to cross the system
(also named DYNAMICAL TIME)

\[ t_{\text{cross}} = \frac{R}{v} \]

\[ = \sqrt{\frac{R^3}{GM}} = \frac{1}{\sqrt{G} \rho} \]
RELAXATION TIME = time necessary for stars in a system to lose completely the memory of their initial velocity

\[ t_{rlx} = n_{cross} \quad t_{cross} = \frac{N}{8 \ln N} \frac{R}{v} \]

with more accurate calculations, based on diffusion coefficients (Spitzer & Hart 1971):

\[ t_{rlx} = 0.34 \frac{\sigma^3}{G^2 m \rho \ln \Lambda} \]
The two expressions are almost equivalent

If we put \( \sigma = \nu = (G N m / R)^{1/2} \)
and \( \rho \propto N m / R^3 \)
and \( \ln \Lambda \sim \ln N \)

\[
\begin{align*}
\tau_{rlx} & \propto \frac{\sigma^3}{G^2 m \rho \ln \Lambda} \\
& \sim \frac{(G N m R^{-1})^{3/2}}{G^2 N m^2 R^{-3} \ln N} \\
& \sim G^{-1/2} N^{1/2} m^{-1/2} R^{3/2} \ln N^{-1} \frac{\nu}{\nu} \\
& \sim \left( \frac{N R}{G m} \right)^{1/2} \nu \ln N^{-1} \frac{R}{\nu} \\
& \sim \frac{N}{\ln N} \frac{R}{\nu}
\end{align*}
\]

multiply and divide per \( \nu \)
Rearrange and substitute again \( \nu \)
Relaxation and thermalization are almost **SYNONYMOUS**!

* **Thermalization:**
  - is one case of relaxation
  - is defined for gas (because needs definition of $T$), but can be used also for stellar system (kinetic extension of $T$)
  - is the **process of particles reaching thermal equilibrium through mutual interactions** (involves concepts of equipartition and evolution towards maximum entropy state)
  - has velocity distribution function: **Maxwellian** velocity

* **Relaxation:**
  - is defined not only for gas
  - is the **process of particles reaching equilibrium through mutual interactions**
Which is the typical $t_{\text{rlx}}$ of stellar systems?

* globular clusters, dense young star clusters, nuclear star clusters (far from SMBH influence radius)
  $R\sim 1-10$ pc, $N\sim 10^3-10^6$ stars, $v\sim 1-10$ km/s
  $t_{\text{rlx}} \sim 10^7-10^{10}$ yr
  → COLLISIONAL

* galaxy field/discs
  $R\sim 10$ kpc, $N\sim 10^{10}$ stars, $v\sim 100-500$ km/s
  $t_{\text{rlx}} \gg$ Hubble time
  → COLLISIONLESS
  described by collisionless Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1,6} \frac{\partial x_i}{\partial t} \frac{\partial f}{\partial x_i} = 0$$
EXAMPLES of COLLISIONAL stellar systems

Globular clusters (47Tuc), by definition
EXAMPLES of COLLISIONAL stellar systems

Nuclear star clusters (MW)
NaCo @ VLT
Genzel+2003

0.4 pc
EXAMPLES of COLLISIONAL stellar systems

Young dense star clusters (Arches, Quintuplet)
EXAMPLES of COLLISIONAL stellar systems

Open clusters, especially in the past (NGC290)
EXAMPLES of COLLISIONAL stellar systems

Embedded clusters, i.e. baby clusters (RCW 38)
NaCo @ VLT
Crowded Places

- galactic nuclei
- globular clusters
- young clusters
- solar neighbourhood

log density (solar masses/pc^3)

log mass (solar masses)
How do star clusters form? BOH

* from giant molecular clouds

* possibly from aggregation of many sub-clumps
GAS SIMULATION
by Matthew Bate (Exeter):
- gas
- SPH
- turbulence
- fragmentation
- sink particles
How do star clusters form?  

* from giant molecular clouds  
* possibly from aggregation of many sub-clumps  
* reach first configuration by VIOLENT RELAXATION (?)
VIOLENT RELAXATION?

- Theory by Linden-Bell in 1967 (MNRAS 136, 101)

- Starts from a problem: galaxy discs and elliptical galaxies are RELAXED (stars follow thermal distribution), even if $t_{rlx} \gg t_{Hubble}$

- IDEAs: (1) there should be another relaxation mechanism (not two-body) efficient on DYNAMICAL timescale (crossing time)
  (2) by 2\textsuperscript{nd} law of thermodynamics: such mechanism must MAXIMIZE entropy
  (3) RELAXATION DRIVER: the POTENTIAL of a newly formed galaxy or star cluster CHANGES VIOLENTELY (on dynamical time)

→ changes in potential must redistribute stellar ENERGY in a CHAOTIC WAY (losing memory of initial conditions)
- **VIOLENT RELAXATION?**  
  - HERE the problems start, because
    * in simulations, it is difficult to disentangle numerical instability from true effects
    * in calculations, there are too many approximations
  Note: there is NOT a complete mathematical formulation of this problem

- Formulation by Linden-Bell distinguishes
  Fine-grained distribution function in phase space
  
  \[
  f(\vec{x}, \vec{y}, t) = \frac{dm(\vec{x}, \vec{y}, t)}{d^3\vec{x} \, d^3\vec{v}}
  \]

  Coarse-grained distribution function in phase space
  
  \[
  F(\vec{x}, \vec{y}, t) = \frac{1}{\Lambda^3 \vec{x} \, \Lambda^3 \vec{v}} \int_{\Lambda^3 \vec{x} \, \Lambda^3 \vec{v}} f(\vec{x}, \vec{y}, t) \, d^3\vec{x} \, d^3\vec{v}
  \]

  and demonstrates that the CORRECT coarse-grained distribution after 1 dynamical time is a thermal distribution (\(=\) Maxwellian), which maximizes entropy.

  **BUT:** this may be wrong..there are cases where it does not work
VIOLENT RELAXATION?
- POSSIBLE SOLUTIONS:
  (1) relaxation is too fast to maximize entropy in all intermediate states (Arad and Lynden-Bell, MNRAS, 361, 385 2005)

  (2) Lynden-Bell is right, but only if the initial system satisfies the virial condition

If not, the system oscillates, ejects mass (by evaporation) and starts gravothermal collapse (by Levin, Pakter & Rizzato 2008, Phys. Rev. E 78, 021130)

\( \chi := \text{deviation from stationary state} \)

See also review by Bindoni & Secco 2008, New Astronomy, 52, 1
How do star clusters form?

* from giant molecular clouds

* possibly from aggregation of many sub-clumps

* reach first configuration by VIOLENT RELAXATION (?)

* after this can be modelled by
  PLUMMER SPHERE
  ISOTHERMAL SPHERE
  LOWERED ISOTHERMAL SPHERE
  KING MODEL

* after reaching first configuration, they become COLLISIONAL and relax through two-body encounters faster than their lifetime (even without mass spectrum and stellar evolution!)

* can DIE by INFANT MORTALITY!!!
INFANT MORTALITY: clusters can die when GAS is removed

DENSE CLUSTERS
- bound
- $< \sim 10^5 \text{ pc}^{-3} \ (\text{coll.})$
- $\sim 10^3 - 10^5$ stars

OPEN CLUSTERS
- loosely bound
- $< 10^4$ stars

ASSOCIATIONS
- unbound
- $< 10^3$ stars

OPEN and DENSE STAR CLUSTERS as SURVIVORS of INFANT MORTALITY: how and with which properties?
INFANT MORTALITY

-DEPENDENCE on SFE : = \frac{M_{\text{star}}}{(M_{\text{star}} + M_{\text{gas}})}

(1) Velocity dispersion from virial theorem before gas removal:

\[ \sigma_0^2 = \frac{G (M_{\text{star}} + M_{\text{gas}})}{R_0} \]

(2) Energy after gas removal (hypothesis of instantaneous gas removal):

\[ E = \frac{1}{2} M_{\text{star}} \sigma_0^2 - \frac{G M_{\text{star}}^2}{R_0} \]

(3) Energy after new virialization:

\[ E = -\frac{G M_{\text{star}}^2}{2R} \]

New cluster size:
- From (2) = (3)

\[ -\frac{G M_{\text{star}}^2}{2R} = \frac{1}{2} M_{\text{star}} \sigma_0^2 - \frac{G M_{\text{star}}^2}{R_0} \]

INFANT MORTALITY

-DEPENDENCE on SFE : $\rho = \frac{M_{\text{star}}}{(M_{\text{star}} + M_{\text{gas}})}$

New cluster size:
- Using (1)

\[
\frac{M_{\text{star}}}{2R} = \frac{1}{2} \left( \frac{M_{\text{star}} + M_{\text{gas}}}{R_0} \right) - \frac{M_{\text{star}}}{R_0}
\]

-Rearranging

\[
R = R_0 \frac{M_{\text{star}}}{M_{\text{star}} + M_{\text{gas}}} \frac{1}{\left(2 \frac{M_{\text{star}}}{M_{\text{star}} + M_{\text{gas}}} - 1\right)}
\]

$R > 0$ only if

\[
\frac{M_{\text{star}}}{M_{\text{star}} + M_{\text{gas}}} > 0.5
\]

INFANT MORTALITY

-DEPENDENCE on SFE: <30% disruption

-DEPENDENCE on t_gas:
  explosive removal: $t_{gas} \ll t_{cross}$
  [smaller systems]
  adiabatic removal: $t_{gas} \sim t_{cross}$
  [dense clusters]

-DEPENDENCE on the (+/-) VIRIAL state of the embedded cluster

-DEPENDENCE on Z: metal poor clusters more compact than metal rich

Hills 1980; Lada & Lada 2003; Bastian & Goodwin 2006; Baumgardt & Kroupa 2007; Bastian 2011; Pelupessy & Portegies Zwart 2011
INFANT MORTALITY

A criterion to infer whether a star cluster is dying or will survive, empirically found by Gieles & Portegies Zwart (2011, MNRAS, 410, L6)

$$\Pi \equiv \frac{Age}{t_{dyn}}$$

$$t_{dyn} \equiv 10 \left( \frac{R_{hl}^3}{GM} \right)^{1/2}$$

$$\Pi > 1$$ surviving star cluster

$$\Pi < 1$$ association  (maybe)
After crossing time, relaxation time, the third important timescale for Collisional (and even collisionless) systems is **DYNAMICAL FRICTION TIMESCALE**

A body of mass $M$, traveling through an infinite & homogeneous sea of bodies (mass $m$) suffers a steady deceleration: the dynamical friction infinite & homogeneous sea: otherwise the body $M$ would be deflected

The sea exerts a force parallel and opposite to the velocity $V_0$ of the body

It can be shown that DYNAMICAL FRICTION TIMESCALE is

\[
    t_{df} = \frac{3}{4 \left(2 \pi\right)^{1/2}} \frac{\sigma^3(r)}{G^2 \ln \Lambda \frac{M \rho(r)}{}}
\]
The basic idea of dynamical friction is as follows:

- The heavy body $M$ attracts the lighter particles.
- When lighter particles approach, the body $M$ has already moved and leaves a local overdensity behind it.
- The overdensity attracts the heavy body (with force $F_d$) and slows it down.
REPETITA IUVANT: Plummer sphere

Isotropic velocity distribution function: \[ f(E) \propto \begin{cases} (-E)^p & \text{if } E < 0 \\ 0 & \text{if } E \geq 0 \end{cases} \]

If \( p=1 \) corresponds to potential \( \phi(r) = -\frac{G M}{(r^2 + a^2)^{1/2}} \)

From Poisson equation \( \nabla^2 \phi = 4\pi G \rho \)

We derive density \[ \rho(r) = \frac{M}{\frac{4}{3} \pi a^3} \frac{1}{\left[1 + \left(\frac{r}{a}\right)^2\right]^{5/2}} \]

and corresponding mass \[ M(r) = \frac{M}{a^3} \frac{r^3}{\left[1 + \left(\frac{r}{a}\right)^2\right]^{3/2}} \]
REPETITA IUVANT: Plummer sphere

\begin{align*}
\rho & \quad [M_\odot \text{ pc}^{-3}] \\
M(r) & \quad [M_\odot]
\end{align*}

\begin{align*}
\text{r [pc]} & \quad \text{r [pc]}
\end{align*}
REPETITA IUVANT: Isothermal sphere

* Why isothermal? From formalism of ideal gas

If \( T = \text{const} \), \( P = \text{const} \times \rho \)

* For politropic equation of state
  is isothermal if \( \gamma = 1 \)

if we assume hydrostatic equilibrium

we derive the potential

\[
\phi = -\kappa \ln \left( \frac{\rho}{\rho_c} \right)
\]

using Poisson's equation we find

\[
\rho(r) = \frac{\sigma^2}{2\pi G r^2}
\]

expressing the constant \( k \) with some physical quantities
REPETITA IUVANT: Isothermal sphere

\[
\rho \equiv \frac{M}{4\pi r^2} \quad \text{and} \quad M(r) \equiv \frac{4}{3} \pi r^3 
\]

where $\rho$ is the density and $M(r)$ is the mass within radius $r$. The graphs depict the variation of density and mass with radius $r$. The data shows a decrease in density with increasing radius in the left graph, and the mass within a sphere increases linearly with radius in the right graph.
REPETITA IUVANT: PROBLEMS of isothermal sphere

1) DENSITY goes to infinity if radius goes to zero

\[ \rho(r) = \frac{\sigma^2}{2 \pi G r^2} \]

2) MASS goes to infinity if radius goes to infinity

\[ M(r) = 4 \pi \int_0^r \rho(r) r^2 \, dr = \frac{2 \sigma^2}{G} r \]
REPETITA: Lowered isothermal sphere and King model

1) King model (also said non-singular isothermal sphere) solves the problem at centre by introducing a CORE

\[ r_0 = \sqrt{\frac{9 \sigma^2}{4 \pi G \rho_0}} \]

with the core, \( \rho \) has a difficult analytical shape, but can be approximated with the singular isothermal sphere for \( r \gg r_0 \)

and with

\[ \rho(r) = \rho_0 \frac{1}{\left[1 + \left(\frac{r}{r_0}\right)^2\right]^{3/2}} \]

For \( r \ll 2 \ r_0 \)
REPETITA: Lowered isothermal sphere and King model

2) Non-singular LOWERED isothermal sphere := all King models where mass is truncated at a certain radius (does not go to infinity)

VELOCITY DISTRIBUTION FUNCTION

\[ f(E) \propto \begin{cases} \kappa \left( e^{-B \cdot E} - e^{-B \cdot E_e} \right) & \text{if } E < E_e \\ 0 & \text{if } E \geq E_e \end{cases} \]

NOTE: VELOCITY DISTRIBUTION FUNCTION is the MAXWELLIAN for isothermal sphere and truncated Maxwellian for lowered non-singular isothermal sphere!!!
References:


* Spitzer L., Dynamical evolution of globular clusters, 1987, Princeton University Press