LECTURES on COLLISIONAL DYNAMICS:

1. RELEVANT TIMESCALES,
FORMATION OF STAR CLUSTERS,
EQUILIBRIUM MODELS
COLLISIONAL/COLLISIONLESS?
Collisional systems are systems where interactions between particles are EFFICIENT with respect to the lifetime of the system
Collisionless systems are systems where interactions are negligible

When is a system collisional/collisionless?

RELAXATION TIMESCALE
Gravity is a LONG-RANGE force → cumulative influence on each star/body of distant stars/bodies is important: often more important than influence of close stars/bodies

Let us consider a IDEALIZED galaxy of N identical stars with mass \( m \), size \( R \) and uniform density
Let us focus on a single star that crosses the system

How long does it take for this star to change its initial velocity completely?, i.e. by

\[
\frac{\delta \vec{v}_\perp}{\vec{v}} \sim 1
\]
Let us assume that our test star passes close to a field star at relative velocity $v$ and impact parameter $b$. The test star and the perturber interact with a force

$$F_\perp = \frac{G m^2}{r^3} \quad r_\perp = \frac{G m^2}{r^2} \cos \theta$$

$$= \frac{G m^2 b}{(x^2 + b^2)^{3/2}} = \frac{G m^2}{b^2} \left[1 + \left(\frac{v \cdot t}{b}\right)^2\right]^{3/2}$$
From Newton's second law \( m \ddot{v}_\perp = F_\perp \)

we get that the perturbation of the velocity integrated over one entire encounter is

\[
\delta v_\perp = \int_{-\infty}^{+\infty} \frac{F_\perp}{m} \, dt = \frac{G \ m}{b^2} \int_{-\infty}^{+\infty} \frac{dt}{\left[1 + \left(\frac{v \ t}{b}\right)^2\right]^{3/2}} = \frac{2 \ G \ m}{b \ v}
\]

\[
dt = \frac{b}{v} \frac{d}{d} \left(\frac{v \ t}{b}\right)
\]

Accel. at closest approach

Force duration

\[
2 \left(\frac{G \ m}{b^2}\right) \left(\frac{b}{v}\right)
\]
Now we account for all the particles in the system:

Surface density of stars in idealized galaxy:

\[
\frac{N}{\pi R^2}
\]

Number of interactions per unit element:

\[
\delta n = \frac{N}{\pi R^2} 2 \pi b \, db
\]

We define

\[
\delta v_{TOT}^2 = \delta n \delta v_\perp^2 = \frac{2 N}{R^2} \left( \frac{2 G m}{b \nu} \right)^2 \, b \, db
\]

And we integrate over all the possible impact parameters...
And we integrate over all the possible impact parameters...

\[ \delta v_{\text{TOT}}^2 = 8 \, N \left( \frac{G \, m}{R \, v} \right)^2 \int_{b_{\text{min}}}^{R} \frac{db}{b} \]

\[ \delta v_{\text{TOT}}^2 = 8 \, N \left( \frac{G \, m}{R \, v} \right)^2 \ln \frac{R}{b_{\text{min}}} \]

* low integration limit: smallest \( b \) to avoid close encounter \( \delta v_\perp \sim v \)

\[ v = \frac{2 \, G \, m}{b_{\text{min}} \, v} \quad \implies \quad b_{\text{min}} = \frac{2 \, G \, m}{v^2} \]

* top integration limit: size \( R \) of the system
And we integrate over all the possible impact parameters...

\[
\delta v^2_{TOT} = 8 \, N \, \left( \frac{G \, m}{R \, v} \right)^2 \ln \left( \frac{R \, v^2}{2 \, G \, m} \right)
\]

Typical speed of a star in a virialized system

\[
N \, m \, v^2 = \frac{G \, (N \, m)^2}{R} \quad \quad \Rightarrow \quad v^2 = \frac{G}{N \, m} \, R
\]

Replacing \( v \)

\[
\frac{\delta v^2_{TOT}}{v^2} = \frac{8 \, \ln N}{N}
\]
Number of crossings of the system for which

\[ \frac{\delta v^2_{\text{TOT}}}{v^2} = 1 \]

\[ v^2 = \frac{N}{8 \ln N} \delta v^2_{\text{TOT}} \]
Number of crossings of the system for which

\[ n_{\text{cross}} \frac{\delta v^2_{\text{TOT}}}{v^2} = 1 \]

\[ n_{\text{cross}} \frac{\delta v^2_{\text{TOT}}}{v^2} = \frac{N}{8 \ln N} \delta v^2_{\text{TOT}} \]
Number of crossings of the system for which \( \frac{\delta u^2_{\text{TOT}}}{u^2} = 1 \)

\[
n_{\text{cross}} = \frac{N}{8 \ln N}
\]

**CROSSING TIME** = time needed to cross the system (also named DYNAMICAL TIME)

\[
t_{\text{cross}} = \frac{R}{v}
\]

\[
= \sqrt{\frac{R^3}{GM}} = \frac{1}{\sqrt{G\rho}}
\]
RELAXATION TIME = time necessary for stars in a system to lose completely the memory of their initial velocity

\[ t_{rlx} = n_{\text{cross}} \quad t_{\text{cross}} = \frac{N}{8 \ln N} \frac{R}{v} \]

with more accurate calculations, based on diffusion coefficients (Spitzer & Hart 1971):

\[ t_{rlx} = 0.34 \frac{\sigma^3}{G^2 m \rho \ln \Lambda} \]
The two expressions are almost equivalent.

If we put $\sigma = v = (G N m / R)^{1/2}$
and $\rho \propto N m / R^3$
and $\ln \Lambda \sim \ln N$.

\[
t_{\text{rlx}} \propto \frac{\sigma^3}{G^2 m \rho \ln \Lambda} \sim \frac{(G N m R^{-1})^{3/2}}{G^2 N m^2 R^{-3} \ln N}
\]

\[
\sim G^{-1/2} N^{1/2} m^{-1/2} R^{3/2} \ln N^{-1} \left(\frac{v}{\nu}\right)
\]

\[
\sim \left(\frac{N R}{G m}\right)^{1/2} v \ln N^{-1} \frac{R}{\nu}
\]

\[
\sim \frac{N R}{\ln N} \frac{R}{\nu}
\]
If we put \( v = (G N m / R)^{1/2} \)

\[
\begin{align*}
t_{rlx} & \sim \frac{N R}{\ln N \nu} \sim \frac{N^{1/2} R^{3/2}}{(G m)^{1/2} \ln N} \\
& \sim \frac{N}{\ln N} \frac{R^{3/2}}{(G N m)^{1/2}} \sim \frac{R^{3/2}}{(m N)^{1/2} R^{3/2}} \\
& \sim \frac{15 \text{ Myr}}{\left( \frac{M_{TOT}}{10^4 M_\odot} \right)^{1/2} \left( \frac{R}{1 \text{ pc}} \right)^{3/2} \left( \frac{1 M_\odot}{m} \right)}
\end{align*}
\]

Portegies Zwart 2006
RELAXATION & THERMALIZATION

Relaxation and thermalization are almost SYNONYMOUS!

* **Thermalization:**
  - is one case of relaxation
  - is defined for gas (because needs definition of $T$), but can be used also for stellar system (kinetic extension of $T$)
  - is the **process of particles reaching thermal equilibrium through mutual interactions** (involves concepts of equipartition and evolution towards maximum entropy state)
  - has velocity distribution function: **Maxwellian** velocity

* **Relaxation:**
  - is defined not only for gas
  - is the **process of particles reaching equilibrium through mutual interactions**
    (but there might be many processes driving to relaxation not only 2-body relaxation)
Which is the typical $t_{\text{rlx}}$ of stellar systems?

* globular clusters, dense young star clusters, nuclear star clusters (far from SMBH influence radius)
  $R \approx 1-10\text{ pc}$, $N \approx 10^3-10^6$ stars, $v \approx 1-10\text{ km/s}$
  $t_{\text{rlx}} \approx 10^7-10^{10}\text{ yr}$
→ COLLISIONAL

* galaxy field/discs
  $R \approx 10\text{ kpc}$, $N \approx 10^{10}$ stars, $v \approx 100-500\text{ km/s}$
  $t_{\text{rlx}} \gg$ Hubble time

→ COLLISIONLESS
  described by collisionless Boltzmann equation

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1,6} \frac{\partial x_i}{\partial t} \frac{\partial f}{\partial x_i} = 0
\]
EXAMPLES of COLLISIONAL stellar systems

Globular clusters (47Tuc), by definition
EXAMPLES of COLLISIONAL stellar systems

Nuclear star clusters (MW)
NaCo @ VLT
Genzel+2003

0.4 pc
EXAMPLES of COLLISIONAL stellar systems

Young dense star clusters (Arches, Quintuplet)
EXAMPLES of COLLISIONAL stellar systems

Open clusters, especially in the past (NGC290)
EXAMPLES of COLLISIONAL stellar systems

Embedded clusters, i.e. baby clusters (RCW 38)
NaCo @ VLT
DENSITY & MASS ORDER OF MAGNITUDES

Crowded Places

- galactic nuclei
- globular clusters
- young clusters
- solar neighbourhood

\[ \text{log density (solar masses/pc}^3) \]
\[ \text{log mass (solar masses)} \]

M. B. Davies, 2002, astroph/0110466
DISTRIBUTION of COLLISIONAL stellar systems in the MILKY WAY

GLOBULAR CLUSTERS ARE A HALO POPULATION
YOUNG and OPEN CLUSTERS ARE A DISC POPULATION

Portegies Zwart, McMillan & Gieles 2010
MAIN PROPERTIES of COLLISIONAL stellar systems in the MILKY WAY

**TOTAL MASS (Msun)**

| Open clusters | Globular clusters | Young dense star clusters |

$t_{\text{relx}} = 10 \text{ Myr} \left( \frac{M_{\text{TOT}}}{3500 M_{\odot}} \right)^{1/2} \left( \frac{r_{\text{hm}}}{1 \text{ pc}} \right)^{3/2}$

Portegies Zwart, McMillan & Gieles 2010
How do star clusters form?  

* from giant molecular clouds  

* possibly from aggregation of many sub-clumps
CLOUD SIMULATION by Matthew Bate (Exeter):

- gas
- SPH
- turbulence
- fragmentation
- sink particles
SCs form from different cores of a molecular cloud.

More filaments than spherical!
How do star clusters form?

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* reach first configuration by VIOLENT RELAXATION (??)
VIOLENT RELAXATION?  

BOH

- Theory by Linden-Bell in 1967 (MNRAS 136, 101)

- Starts from a problem: galaxy discs and elliptical galaxies are RELAXED (stars follow thermal distribution), even if $t_{rlx} \gg t_{Hubble}$

- IDEAs: (1) there should be another relaxation mechanism (not two-body) efficient on **DYNAMICAL timescale** (crossing time)

  (2) by 2$^{nd}$ law of thermodynamics: such mechanism must MAXIMIZE entropy

  (3) RELAXATION DRIVER: the **POTENTIAL** of a newly formed galaxy or star cluster **CHANGES VIOLENTELY** (on dynamical time)

→ changes in potential must redistribute stellar ENERGY in a **CHAOTIC WAY** (losing memory of initial conditions)
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* after this can be modelled by distribution functions
  PLUMMER SPHERE
  ISOTHERMAL SPHERE
  LOWERED ISOTHERMAL SPHERE
  KING MODEL

* after reaching first configuration, they become COLLISIONAL and relax through two-body encounters faster than their lifetime (even without mass spectrum and stellar evolution!)

* can DIE by INFANT MORTALITY!!!
INFANT MORTALITY: clusters can die when GAS is removed

OPEN and DENSE STAR CLUSTERS as SURVIVORS of INFANT MORTALITY: how and with which properties?

DENSE CLUSTERS
- bound
- $n \sim 10^{3-5} \text{ pc}^{-3} \text{ (coll.)}$
- $\sim 10^{3-6} \text{ stars}$

OPEN CLUSTERS
- loosely bound
- $\sim 10^{3-4} \text{ stars}$

ASSOCIATIONS
- unbound
- $<10^3 \text{ stars}$

OPEN and DENSE STAR CLUSTERS as SURVIVORS of INFANT MORTALITY: how and with which properties?
INFANT MORTALITY: clusters can die when GAS is removed

FORMATION of STAR CLUSTERS: basic concepts

1 * giant molecular cloud: $10^{5-6} \, M_\odot$ of molecular gas, mainly $H_2$, in $\sim10$ pc radius, at 10-100 K
INFANT MORTALITY: clusters can die when GAS is removed

FORMATION of STAR CLUSTERS: basic concepts

1 * giant molecular cloud

2 * gas cools down and compresses → protostars form
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how many SCs survive gas evaporation?
INFANT MORTALITY:= clusters can die when GAS is removed

Intuitive argument:
$|W_0| = G \frac{(M_{\text{gas}} + M_{\text{star}})^2}{R}$

If $M_{\text{star}}$ large with respect to $M_{\text{gas}}$, cluster remains bound
INFANT MORTALITY: clusters can die when GAS is removed

Intuitive argument:

\[ |W_0| = G \frac{(M_{\text{gas}} + M_{\text{star}})^2}{R} \]

If \( M_{\text{star}} \) large with respect to \( M_{\text{gas}} \), cluster remains bound.

If \( M_{\text{star}} \) small with respect to \( M_{\text{gas}} \), then cluster becomes unbound.
INFANT MORTALITY

-DEPENDENCE on SFE : \( \frac{M_{\text{star}}}{(M_{\text{star}} + M_{\text{gas}})} \)

(1) Velocity dispersion from virial theorem before gas removal:
\[
\sigma_0^2 = \frac{G (M_{\text{star}} + M_{\text{gas}})}{R_0}
\]

(2) Energy after gas removal (hypothesis of instantaneous gas removal):
\[
E = \frac{1}{2} M_{\text{star}} \sigma_0^2 - \frac{G M_{\text{star}}^2}{R_0}
\]

(3) Energy after new virialization:
\[
E = - \frac{G M_{\text{star}}^2}{2 R}
\]

New cluster size:
- From (2) = (3)
\[
- \frac{G M_{\text{star}}^2}{2 R} = \frac{1}{2} M_{\text{star}} \sigma_0^2 - \frac{G M_{\text{star}}^2}{R_0}
\]

INFANT MORTALITY

-DEPENDENCE on SFE : 

\[ \frac{M_{\text{star}}}{M_{\text{star}} + M_{\text{gas}}} \]

New cluster size:
- Using (1)

\[ \frac{M_{\text{star}}}{2R} = \frac{1}{2} \left( \frac{M_{\text{star}} + M_{\text{gas}}}{R_0} \right) - \frac{M_{\text{star}}}{R_0} \]

-Rearranging

\[ R = R_0 \frac{M_{\text{star}}}{M_{\text{star}} + M_{\text{gas}}} \left( \frac{1}{2 \frac{M_{\text{star}}}{M_{\text{star}} + M_{\text{gas}}} - 1} \right) \]

\[ R > 0 \text{ only if } \frac{M_{\text{star}}}{M_{\text{star}} + M_{\text{gas}}} > 0.5 \]

INFANT MORTALITY

- DEPENDENCE on SFE: <30% disruption

- DEPENDENCE on $t_{gas}$:
  explosive removal: $t_{gas} \ll t_{cross}$
  [smaller systems]

  adiabatic removal: $t_{gas} \sim t_{cross}$
  [dense clusters]

- DEPENDENCE on the (+/-) VIRIAL state
  of the embedded cluster

- DEPENDENCE on Z: metal poor clusters more compact than metal rich

Hills 1980; Lada & Lada 2003; Bastian & Goodwin 2006;
Baumgardt & Kroupa 2007; Bastian 2011;
Pelupessy & Portegies Zwart 2011
INFANT MORTALITY

A criterion to infer whether a star cluster is dying or will survive, empirically found by Gieles & Portegies Zwart (2011, MNRAS, 410, L6)

\[ \Pi \equiv \frac{Age}{t_{dyn}} \]

\[ t_{dyn} \equiv 10 \left( \frac{R_{hl}^3}{GM} \right)^{1/2} \]

\[ \Pi > 1 \text{ surviving star cluster} \]
\[ \Pi < 1 \text{ association (maybe)} \]
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**DISTRIBUTION FUNCTION or PHASE SPACE DENSITY**

\[ f(\vec{x}, \vec{v}, t) \, d^3\vec{x} \, d^3\vec{v} \]

Number of stars in the infinitesimal volume \( d^3x \) and in the small range of velocities \( d^3v \)

**DISTRIBUTION FUNCTIONS ARE WELL DEFINED ONLY FOR COLLISIONLESS SYSTEMS!!**

Because they can be CONTINUOUS only if potential is smooth

BUT if system is COLLISIONAL potential is not smooth, particles jump from one side to the other of the phase space!

For a short time even a collisional system can be defined by a distribution function (not correct but useful in practice)

Then 2-body relaxation produces jumps and collisional system passes from one equilibrium to another
DISTRIBUTION FUNCTION or PHASE SPACE DENSITY

* Equations of motion in the phase space using distribution functions can be expressed with collisionless BOLTZMANN EQUATION (CBE)

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1,6} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_i} = 0
\]

-same as continuity equation for fluids: valid only if no jumps

JEANS theorem: any steady-state solution of the CBE is a function of the integrals of motion and any function of the integrals of motion is a steady-state solution of the CBE

* Poisson Vlasov equation describes relation between gravity force and its Sources (same as Gauss)

\[ \nabla^2 \phi = 4\pi G \rho \]

* Often potentials and energies are given as

RELATIVE potential: \[ \Psi = -\Phi + \Phi_0 \]

RELATIVE energy: \[ \mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}v^2 \]
Plummer sphere
Plummer sphere

Isotropic velocity distribution function: \( f(E) \propto \begin{cases} 
(-E)^p & \text{if } E < 0 \\
0 & \text{if } E \geq 0
\end{cases} \)

if \( p=1 \) corresponds to potential \( \phi(r) = -\frac{G M}{(r^2 + a^2)^{1/2}} \)

From Poisson equation \( \nabla^2 \phi = 4 \pi G \rho \)

We derive density \( \rho(r) = \frac{M}{\frac{4}{3} \pi a^3} \frac{1}{\left[ 1 + \left( \frac{r}{a} \right)^2 \right]^{5/2}} \)

and corresponding mass \( M(r) = \frac{M}{a^3} \frac{r^3}{\left[ 1 + \left( \frac{r}{a} \right)^2 \right]^{3/2}} \)
Isothermal sphere

- Graph 1: Density $\rho$ vs. radius $r$ in pc.
- Graph 2: Mass $M(r)$ vs. radius $r$ in pc.
**Isothermal sphere**

* Why isothermal? From formalism of ideal gas

\[ P = \frac{k_B}{\mu m_p} \rho T \]

If \( T = \text{const} \) \[ P = \text{const} \times \rho \]

* For politropic equation of state is isothermal if \( \gamma = 1 \)

If we assume hydrostatic equilibrium

\[ \frac{d\phi}{dr} = \frac{1}{\rho} \frac{dP}{dr} = -\frac{\kappa}{\rho} \frac{d\rho}{dr} \]

we derive the potential

\[ \phi = -\kappa \ln \left( \frac{\rho}{\rho_c} \right) \]

using Poisson's equation we find

\[ \rho(r) = \frac{\sigma^2}{2\pi G} r^{-2} \]

expressing the constant \( k \) with some physical quantities
PROBLEMS of isothermal sphere

1) DENSITY goes to infinity if radius goes to zero

\[ \rho(r) = \frac{\sigma^2}{2\pi G r^2} \]

2) MASS goes to infinity if radius goes to infinity

\[ M(r) = 4\pi \int_0^r \rho(r) r^2 \, dr = \frac{2\sigma^2}{G} r \]
Non-singular isothermal sphere or King model

1) King model (also said non-singular isothermal sphere) solves the problem at centre by introducing a CORE

\[ \tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{r} = \frac{r}{r_0}, \quad r_0 = \sqrt{\frac{9 \sigma^2}{4 \pi G \rho_0}} \]

\( r_0 \) is the radius at which the projected density falls to ~half

with the core, \( \rho \) has a difficult analytical shape, but can be approximated with the singular isothermal sphere for \( r >> r_0 \)
and with

\[ \rho(r) = \rho_0 \frac{1}{\left[ 1 + \left( \frac{r}{r_0} \right)^2 \right]^{3/2}} \]

for \( r << 2 r_0 \)
Lowered non-singular isothermal sphere or lowered King model

2) LOWERED non-singular isothermal sphere := all King models where mass is truncated at a certain radius (does not go to infinity)

VELOCITY DISTRIBUTION FUNCTION:

\[ f(E) \propto \begin{cases} \kappa \left( e^{-BE} - e^{-BE_e} \right) & \text{if } E < E_e \\ 0 & \text{if } E \geq E_e \end{cases} \]

DENSITY EXPRESSION:

\[ \rho_K(\Psi) = \rho_1 \left[ \exp\left( \frac{\Psi}{\sigma^2} \right) \text{erf}\left( \frac{\sqrt{\Psi}}{\sigma} \right) - \sqrt{\frac{4\Psi}{\pi \sigma^2}} \left( 1 + \frac{2\Psi}{3 \sigma^2} \right) \right] \]

Relative potential
Error function
Lowered non-singular isothermal sphere or lowered King model

2) LOWERED non-singular isothermal sphere := all King models where mass is truncated at a certain radius (does not go to infinity)

TIDAL RADIUS \((r_t)\):
Radius at which \(\Psi = 0\) (and \(\rho = 0\))

\[ 0 = \Psi(r_t) = -\Phi(r_t) + Const \]

\(\Phi(0) = \Phi(r_t) - \Psi(0)\)

\[ W_0 = \Psi(0)/\sigma^2 \]

\[ c = \log_{10}(r_t/r_0) \]

Most important parameters of the King model.
Lowered non-singular isothermal sphere or lowered King model

$\rho/\rho_0$ versus $r/r_0$

$c$ versus $W_0$

Binney & Tremaine 1987
Lowered non-singular isothermal sphere or lowered King model

2) LOWERED non-singular isothermal sphere := all King models where mass is truncated at a certain radius (does not go to infinity)

NOTE: VELOCITY DISTRIBUTION FUNCTION is the MAXWELLIAN for isothermal sphere and truncated Maxwellian for lowered non-singular isothermal sphere!!!

\[
f(v) \propto v^2 e^{-m v^2 / (2 k_b T)}
\]
After crossing time, relaxation time, the third important timescale for Collisional (and even collisionless) systems is DYNAMICAL FRICTION TIMESCALE

A body of mass $M$, traveling through an infinite & homogeneous sea of bodies (mass $m$) suffers a steady deceleration: the dynamical friction

The sea exerts a force parallel and opposite to the velocity $V_0$ of the body

It can be shown that DYNAMICAL FRICTION TIMESCALE is

$$t_{df} = \frac{3}{4 (2 \pi)^{1/2}} \frac{\sigma^3(r)}{M \rho(r)} G^2 \ln \Lambda$$
BASIC IDEA:

The heavy body $M$ attracts the lighter particles. When lighter particles approach, the body $M$ has already moved and leaves a local overdensity behind it. The overdensity attracts the heavy body (with force $F_d$) and slows it down.
DYNAMICAL FRICTION vs 2-body RELAXATION:

Dynamical friction timescale:

\[
    t_{df} = \frac{3}{4 \ (2 \pi)^{1/2}} \frac{G^2 \ ln \ \Lambda}{M \ \rho(r)} \frac{\sigma^3(r)}{\rho(r)}
\]

Two-body relaxation timescale:

\[
    t_{rlx} = 0.34 \ \frac{\sigma^3}{G^2 \ m \ \rho \ ln \ \Lambda}
\]

They are relatives..

\[
    t_{df} \sim \frac{m}{M} \ t_{rlx}
\]
References:


* Spitzer, Dynamical evolution of globular clusters, 1987, Princeton University Press

